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Pole Placement in Optimal Regulator by Continuous Pole-Shifting

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This paper presents a method of pole placement in the linear quadratic regulator and its application to the flight control system design. There are three features in the proposed method. First, a weighting matrix that gives desired closed-loop pole locations is obtained by solving a set of differential equations. They are derived from the characteristic equation of the Hamilton matrix. Second, poles can be placed exactly at the desired positions and arbitrarily except the symmetry of complex conjugate poles with respect to the real axis. Third, it is a diagonal weighting matrix that is obtained by the proposed method. The third feature makes output regulation with desired pole locations possible. After demonstrating the effectiveness of the method through a simple literature example, it is then applied to the F-4 aircraft's lateral dynamic model.

I. Introduction

THE optimal regulator, or linear quadratic regulator (LQR), is a typical state-space design method. It has several features. For example, by imposing penalty on state variables and control inputs in a quadratic performance index, one can investigate tradeoff between system performance and control efforts. Besides, it is well known that the LQR has infinite gain margin and 60-deg phase margin for single-input systems. However, by this technique one cannot assign closed-loop pole locations. Since pole locations have a large effect on time-response characteristics, such as overshoot, rise time, etc., it is desirable to assign pole positions in the LQR design.

Pole placement in the LQR is a problem associated with the selection of appropriate weighting matrices, when a set of desired closed-loop eigenvalues (CLEs) are known. This problem has been studied by many researchers since early 1970s, and many papers[1–13] have been published in this research area. Among them, the study of Hayase[1] provides a relation between the weighting matrix and the feedback gains for a single-input system in controllable canonical form. Recently, Ohta et al.[2] showed a simple derivation of the relation. Meanwhile, for multi-input systems, Saif[3] established a simple method using aggregation to compute a weighting matrix by which arbitrary and exact pole placement can be attained in a general form. He integrated the previously proposed three approaches.[4–6] All of the aforementioned methods for multi-input systems follow Solheim's approach for placing imaginary parts as well as for placing real parts using the mirror image property.[7] Another approach, which is completely different from the preceding approaches, is that of Graupe. In his method, first, differential equations of a weighting matrix with respect to the CLEs are derived, and then a weighting matrix is found as a solution of an optimization problem, wherein the difference between CLEs and assigned eigenvalues is made as small as possible.[8] However, it is complicated and does not seem practical. Besides the preceding approaches, there are many approaches such as assigning eigenvalues with prescribed degree of stability,[9] placing eigenvalues into a specified region,[10,11] and achieving pole placement asymptotically.[12,13]

In this paper, we propose a method of pole placement that is very different from the previously proposed methods. Whereas Graupe's differential equations are derived from the Riccati equation, we derived the differential equations from the characteristic equation of the Hamilton matrix. Since these equations do not include solutions of the Riccati equation, they are simple and one is left with simple integration to find an appropriate weighting matrix, assigning the desired CLEs. Unlike Saif's method, real and imaginary parts of complex conjugate eigenvalues do not need shifting separately. In addition, the weighting matrix we provide is diagonal. By taking advantage of this feature, it is possible to achieve optimal output regulation with desired CLEs.

To show the effectiveness of the proposed method through computer simulation, first, it was applied to a simple third-order system, which was also used by Amin and Saif. Then, to illustrate output regulation, a design problem for a lateral-directional flight control system has been carried out with the F-4 aircraft model.[12,13]

An outline of this paper is as follows. Following the Introduction, in Sec. II, the LQR problem is defined, and the relevant Hamilton matrix and its characteristic equation are shown. In Sec. III, the differential equations are derived from the characteristic equation by using Taylor series expansion, and some remarks about practical computation are made. In Sec. IV, the proposed method is extended to the output regulation problem. In Sec. V, simulation results are shown and evaluated. Finally, concluding remarks are given in Sec. VI.

II. Weighting Matrix and Closed-Loop Eigenvalues

LQR Problem

Let us consider a problem of finding an optimal control law that minimizes the performance index of Eq. (2) for a linear system described by Eq. (1),

$$\dot{x} = Ax + Bu \tag{1}$$

$$J = \int_0^\infty \left(x^T Q_x x + u^T R u\right) \mathrm{d}t \tag{2}$$

In Eq. (1), $x \in R^n$ is a state vector, $u \in R^r$ is a control vector, and $A \in R^{n\times n}$ and $B \in R^{n\times r}$ are constant matrices. In Eq. (2), $Q_x \in R^{n\times n}$ is a nonnegative definite matrix and $R \in R^{r\times r}$ a positive definite one. In the following, Q_x is assumed to be a diagonal matrix, i.e., $Q_x = \mathrm{diag}\{q_1, \ldots, q_n\}$. Since pole placement is considered in this paper, (A, B) is assumed to be a controllable pair.

An optimal control law is given by

$$u = -R^{-1}B^T P x \tag{3}$$

where P is a positive definite matrix that satisfies the following algebraic Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q_x = 0 \tag{4}$$

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For the closed-loop system to be stable, (A, C_x) is assumed to be a detectable pair, where C_x is an appropriate matrix that satisfies $Q_x = C_x^T C_x$ and has the same rank as Q_x.[9]

Characteristic Equation of a Hamilton Matrix

The following matrix, $H \in R^{2n\times 2n}$, is called a Hamilton matrix:

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q_x & -A^T \end{bmatrix} \tag{5}$$

The characteristic polynomial for H is given by[3,4]

$$\det(\lambda I_{2n} - H) = \det(\lambda I_n - A)\det[I_n - Q_x S(\lambda)]\det\left(\lambda I_n + A^T\right) \tag{6}$$

where $\det(\cdot)$ stands for a determinant, and I_i ($i = n$ or $2n$) an $i \times i$ identity matrix. The term $S(\lambda)$ is defined by

$$S(\lambda) = (\lambda I_n - A)^{-1} BR^{-1}B^T\left(\lambda I_n + A^T\right)^{-1} \tag{7}$$

Generally, the following properties concerning eigenvalues of H are known[14]:

1) Stable eigenvalues of H are CLEs of the LQR.
2) CLEs are different from open-loop ones in general, if $Q_x > 0$.
3) If λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of H.

From the preceding properties, it follows that $\det(\lambda I_n - A) \neq 0$ and $\det(\lambda I_n + A) = \det(\lambda I_n + A^T) \neq 0$. Therefore the characteristic equation of H can be reduced further as follows:

$$\det[I_n - Q_x S(\lambda)] = 0 \tag{8}$$

The second preceding property is not always true, if $Q_x \geq 0$. The case where some open-loop eigenvalues (OLEs) are included in the CLEs will be discussed later.

In the following discussion, let us keep R as a constant matrix, while performing iterations on Q_x. Given $\lambda = \lambda_i$ ($i = 1, \ldots, n$), Eq. (8) can be regarded as n simultaneous equations with n unknowns q_i. Hence, the following equations,

$$\det[I_n - Q_x S(\lambda_i)] = 0, \qquad i = 1, 2, \ldots, n \tag{9}$$

determine q_i. Solheim solved the equations by transforming the A matrix in Eq. (1) into a diagonal one. However, to obtain a weighting matrix for the original system of Eq. (1), inverse transformation is required, so that the resultant weighting matrix becomes a nondiagonal one. This is the case with Saif's method.

III. Pole Placement Using Differential Equations

Derivation of Differential Equations

Let us define $f(Q_x, \lambda_i)$ as

$$f(Q_x, \lambda_i) = \det[I_n - Q_x S(\lambda_i)] \tag{10}$$

Let the nominal weighting matrices be Q_x and R, and let the CLEs for those weighting matrices be $\Lambda = [\lambda_1, \ldots, \lambda_n]^T$. Let us consider small perturbations $\Delta Q_x (= \mathrm{diag}\{\Delta q_i\})$ and $\Delta\Lambda = [\Delta\lambda_1, \ldots, \Delta\lambda_n]^T$ ($i = 1, \ldots, n$) from the nominal values Q_x and Λ, respectively. By applying Taylor series expansion to $f(Q_x, \lambda_i)$ around the nominals Q_x and λ_i and taking Eq. (9) into account, the following equations are obtained using first-order approximation:

$$\Delta f(Q_x, \lambda_i) = \frac{\partial f_i}{\partial \lambda_i}\Delta\lambda_i + \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\Delta q_j, \qquad i = 1, 2, \ldots, n \tag{11}$$

where f_i indicates $f(Q_x, \lambda_i)$ in Eq. (10).

Let the desired CLEs be $\Lambda^* = [\lambda_1^*, \ldots, \lambda_n^*]^T$, and let the initial CLEs for an initial weighting matrix Q_{x0}, which is given appropriately, be $\Lambda_0 = [\lambda_{01}, \ldots, \lambda_{0n}]^T$. The deviations $\Delta\lambda_i$ can then be defined as

$$\Delta\lambda_i = \frac{\lambda_i^* - \lambda_{0i}}{N} \qquad i = 1, 2, \ldots, n \tag{12}$$

where N is a large integer. Further defining $\Delta\lambda = \min_i |\Delta\lambda_i|$, k_i can be defined as

$$k_i = \Delta\lambda_i / \Delta\lambda \tag{13}$$

With the preceding definitions, Eq. (11) can be rewritten as

$$\Delta f_i = \frac{\partial f_i}{\partial \lambda_i} k_i \Delta\lambda + \sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j}\Delta q_j \qquad i = 1, 2, \ldots, n \tag{14}$$

For Eq. (8) to hold with $(Q_x + \Delta Q_x, \lambda_i + \Delta\lambda_i)$ instead of (Q_x, λ_i), the perturbations must satisfy the following equation:

$$\Delta f(Q_x, \lambda_i) = 0 \tag{15}$$

Then Eqs. (14) and (15) yield

$$\frac{\partial f}{\partial q^T}\Delta q = -g\Delta\lambda \tag{16}$$

where $f = [f_1, \ldots, f_n]^T$, $q = [q_1, \ldots, q_n]^T$, and $g = [(\partial f_1/\partial\lambda_1)k_1, \ldots, (\partial f_n/\partial\lambda_n)k_n]^T$.

From Eq. (16) the following equation is obtained:

$$\frac{\Delta q}{\Delta\lambda} = -\left[\frac{\partial f}{\partial q^T}\right]^{-1} g \tag{17}$$

Integrating the ordinary differential equation defined in Eq. (17) from $\Lambda = \Lambda_0$ to Λ^*, we get $Q_x^* [= \mathrm{diag}(q_i^*)]$ for the desired CLEs Λ^*. The right-hand side of Eq. (17) can be computed as follows.

Equation (18) gives differential of determinant for a matrix $X(t)$,[15]

$$\frac{d\{\det[X(t)]\}}{dt} = \mathrm{tr}\left\{\frac{X(t)}{dt}\mathrm{adj}[X(t)]\right\} \tag{18}$$

where $\mathrm{tr}[\cdot]$ indicates a trace and $\mathrm{adj}(\cdot)$ an adjoint matrix. By defining $X = I_n - Q_x S(\lambda_i)$ and using Eq. (18), the derivatives in Eq. (17) are obtained as

$$\frac{\partial f_i}{\partial \lambda_i} = \mathrm{tr}\left[\frac{\partial X}{\partial \lambda_i}\mathrm{adj}(X)\right] \tag{19}$$

$$\frac{\partial f_i}{\partial q_j} = \mathrm{tr}\left[\frac{\partial X}{\partial q_j}\mathrm{adj}(X)\right] \tag{20}$$

where

$$\frac{\partial X}{\partial \lambda_i} = Q_x(\lambda_i I_n - A)^{-1}\left[(\lambda_i I_n - A)^{-1}F + F\left(\lambda_i I_n + A^T\right)^{-1}\right]\left(\lambda_i I_n + A^T\right)^{-1} \tag{21}$$

$$\frac{\partial X}{\partial q_j} = -Z_{jj}S(\lambda_i) \tag{22}$$

and $F = BR^{-1}B^T$. Here, Z_{jj} is an $n \times n$ matrix whose elements are zero except for the (j, j) element, which is one.

For numerical computation, let us make the following remarks.

Remark 1 (Integration Using Real Number Only)

For complex conjugate eigenvalues, Eq. (17) can be integrated using only real numbers by applying a linear transformation to $[\partial f_i/\partial q^T]$ and g. For example, suppose that λ_i and λ_{i+1} are complex conjugate eigenvalues. Then by a linear transformation, real parts of $[\partial f_i/\partial q^T]$ and $[\partial f_i/\partial \lambda_i^T]k_i$ are put in the ith row of $[\partial f_i/\partial q^T]$ and g, respectively. The imaginary parts are put in the $i+1$th row in the same way.

Remark 2 (Pairing Eigenvalues)

Case A (Number of Real Eigenvalues in the Spectrum Λ_0 and Λ^* Are Same)

Whenever possible, combine the existing real eigenvalue λ_{0i} with the desired real eigenvalue λ_i^* for shifting process. This procedure automatically pairs the existing complex conjugate eigenvalues (λ_{0i}, λ_{0i+1}) with the desired complex conjugate eigenvalues (λ_i^*, λ_{i+1}^*). However, in the integration process, it is important to maintain the pair consistency throughout the shifting.

Case B (Number of Real Eigenvalues in the Spectrum Λ_0 and Λ^* Are Different)

Suppose that one wants to shift two real eigenvalues λ_0 to a pair of complex conjugate locations λ^*. Then, first shift the two real λ_0 to one repeated eigenvalue, and then shift it further to λ^*. If initial eigenvalues are complex conjugate and desired ones are real, then take the opposite process. In other words, the continuity of eigenvalues is maintained in the sense of root locus.

Remark 3 (Step Size in Numerical Integration)

In practical computation, one may prefer an appropriately chosen integral step size h and the steps of $N = \max_i |\lambda_i - \lambda_{0i}|/h$ to a large number of steps N and the step size $\Delta\lambda = \min_i |\Delta\lambda_i|$. Accordingly k_i is then defined as $k_i = (\lambda_i^* - \lambda_{0i})/Nh$. If λ_{0i} and λ_i^* are complex conjugate eigenvalues, then let N be $\max_i\{|\mathrm{Re}(\lambda_i^* - \lambda_{0i})|, |\mathrm{Im}(\lambda_i^* - \lambda_{0i})|\}/h$.

Remark 4 (Avoidance of Singularity)

Case A (By Appropriate Choice of the Initial Weighting Matrix)

Choose Q_{x0} such that the CLEs shifted from Λ_0 to Λ^* do not pass through the OLEs. The reason is that $S(\lambda)$ in Eq. (7) becomes singular, if λ agrees with an OLE. To find an appropriate initial weighting matrix, trial and error may be required.

Case B (By Changing the OLEs)

If it is inevitable for CLEs to pass through the OLEs, it is sometimes helpful to regard the closed-loop system designed for Q_{x0} as a new open-loop system. By applying the proposed method to the new system, a weighting matrix is obtained, and then it is added to Q_{x0} to find Q_x^* for the original open-loop system. An example will be shown later. If part of the desired CLEs includes the OLEs, the problem of singularity of $S(\lambda)$ can be resolved in the same way. In this case, however, although the eigenvalues to be changed are part of the OLEs, all of the OLEs have to be shifted to avoid the singularity, unlike most of the methods referred to such as Saif's.

Case C (Repeated Eigenvalues)

Although in Remark 1 we mentioned using repeated eigenvalues as intermediate points in the pole shifting, the proposed method does not allow the closed-loop system to have repeated eigenvalues. The reason is that in such a case, the corresponding row of the derivative matrix in both sides of Eq. (16) becomes zero. To overcome this problem, integration is stopped just before the root locus reaches the repeated eigenvalues, and in case B in remark 1 integration is resumed at the point just after those eigenvalues on the specified root locus. For instance, in the case of shifting from complex conjugate eigenvalues to two real ones, integration is stopped at $(\alpha+j\varepsilon, \alpha-j\varepsilon)$ and resumed at $(\alpha+\varepsilon, \alpha-\varepsilon)$ with the weighting matrix at $(\alpha+j\varepsilon, \alpha-j\varepsilon)$, where α is a repeated eigenvalue. This process causes discontinuity of the root locus that leads to errors in the resulting weighting matrix. However, as shown in a numerical example later, by giving ε a small number, the CLEs are placed at positions so close to the desired ones that there is no problem in practice.

Remark 5 (Uniqueness of the Solution)

The diagonal elements of the weighting matrix are uniquely determined for specified CLEs, regardless of the initial weighting matrix Q_{x0}, although it was observed from only numerical examples.

IV. Extension to Output Regulation

Let us consider an output equation,

$$y = C/x \tag{23}$$

where $y(\in R^m)$ is an output vector and $C(\in R^{m\times n})$ is a constant matrix. Instead of Eq. (2), a cost function is defined here as

$$J = \int_0^\infty \left(y^T Q_y y + u^T R u\right) \mathrm{d}t \tag{24}$$

where $Q_y(\in R^{m\times m})$ is a nonnegative definite matrix and assumed to be a diagonal matrix, i.e., $Q_y = \mathrm{diag}\{q_1, \ldots, q_m\}$. For the closed-loop system to be stable, (A, C_y) is assumed to be a detectable pair, where C_y is an appropriate matrix that satisfies $C^T Q_y C = C_y^T C_y$ and has the same rank as Q_y.[9]

First, let us assume $C = [I_m \quad 0]$, where I_m is an $m \times m$ identity matrix. In this case, the weighting matrix with respect to the state variables can be written as

$$Q_x = C^T Q_y C = \mathrm{diag}\{q_1, \ldots, q_m, 0, \ldots, 0\} \tag{25}$$

In Eq. (25), $q_1, \ldots, q_m$ are adjusted to place m CLEs at the desired positions. While shifting the eigenvalues, other diagonal elements are kept 0, i.e., $q_{m+1}, \ldots, q_n = 0$, which means that $\Delta q_{m+1}, \ldots, \Delta q_n = 0$. Hence, Eq. (16) can be written as

$$\frac{\partial f_{nz}}{\partial q_{nz}^T} \Delta q_{nz} = -g_{nz} \Delta\lambda \tag{26}$$

$$\frac{\partial f_z}{\partial q_{nz}^T} \Delta q_{nz} = -\mathrm{diag}\left(\frac{\partial f_i}{\partial \lambda_i}\right) \Delta\Lambda_z \qquad i = m+1, \ldots, n \tag{27}$$

where $f_{nz} = [f_1, \ldots, f_m]^T$, $q_{nz} = [q_1, \ldots, q_m]^T$, $g_{nz} = [(\partial f_1/\partial\lambda_1) k_1, \ldots, (\partial f_n/\partial\lambda_n) k_n]^T$, $f_z = [f_{m+1}, \ldots, f_n]^T$, and Δq_{nz} and $\Delta\Lambda_z$ are deviations of q_{nz} and $\Lambda_z = [\lambda_{m+1}, \ldots, \lambda_n]^T$, respectively. The elements in $\Lambda_{nz} = [\lambda_1, \ldots, \lambda_m]^T$ are the eigenvalues to be assigned, and those in Λ_z are the eigenvalues not to be assigned.

From Eq. (26), the following equation is obtained:

$$\frac{\Delta q_{nz}}{\Delta\lambda} = -\left[\frac{\partial f_{nz}}{\partial q_{nz}^T}\right]^{-1} g_{nz} \tag{28}$$

Given the desired CLEs Λ_{nz}^* and an initial weighting matrix Q_{y0} (or q_{nz0}), which determines initial CLEs Λ_{nz0}, integrating Eq. (28) from Λ_{nz0} to Λ_{nz}^* yields q_{nz}^* that achieves the desired closed-loop pole placement. Note that, as seen from Eqs. (26) and (27), all of the CLEs are not assigned; namely, as many eigenvalues as the outputs can only be specified. By dividing both sides of Eq. (27) by $\Delta\lambda$ and substituting Eq. (28), differential equations for Λ_z are obtained as

$$\frac{\Delta\Lambda_z}{\Delta\lambda} = \left[\mathrm{diag}\left(\frac{\partial f_i}{\partial\lambda_i}\right)\right]^{-1} \frac{\partial f_z}{\partial q_{nz}^T} \left[\frac{\partial f_{nz}}{\partial q_{nz}^T}\right]^{-1} g_{nz} \qquad i = m+1, \ldots, n \tag{29}$$

The CLEs other than Λ_{nz} vary following Eq. (29). But, in practice, there is no need to solve the equation. Once q_{nz} are obtained from Eq. (28), Λ_z are found by computing the CLEs for Q_x given by Eq. (25). However, this means that we cannot predict the behavior of Λ_z. Hence, if Q_x becomes indefinite, the nonassignable eigenvalues may go into the right half-plane; as a result, the closed-loop system may become unstable. In the output regulation case, Λ_z must be checked, if Q_x is indefinite.

The preceding approach can be extended to the case of a general C matrix, which is assumed to have the full rank, i.e., rank $(C) = m$. Let us define a new state vector ξ as

$$\xi = [y^T \quad z^T]^T \tag{30}$$

where z is a vector defined by $z = Dx$; $D(\in R^{(n-m)\times n})$ is a proper matrix that makes $[C^T, D^T]^T$ nonsingular. By defining $E = [C^T, D^T]^T$, Eq. (30) can also be written as

$$\xi = Ex \tag{31}$$

Using ξ, Eq. (1) can be rewritten as

$$\dot{\xi} = A'\xi + B'u \tag{32}$$

where $A' = EAE^{-1}$ and $B' = EB$. By applying the proposed method to the transformed system Eq. (32), a desired weighting matrix $Q_\xi = \mathrm{diag}\{q'_1, \ldots, q'_m, 0, \ldots, 0\}$ is obtained for Λ^*_{nz}. The weighting matrix for the original system of Eq. (1) is given by

$$Q_x = E^T Q_\xi E = C^T Q_y C \tag{33}$$

where $Q_y = \mathrm{diag}\{q'_1, \ldots, q'_m\}$. The term Q_y does not depend on the selection of D, which is proven in the Appendix. Although the diagonal elements of Q_ξ corresponding to the states z were zero in the preceding definition of Q_ξ, they can be chosen as nonzero values. If those elements are nonzero, Q_y depends on D.

V. Simulation

Simulation 1

First, let us consider the linear system whose matrices A and B in Eq. (1) are given by

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \\ -2 & -2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

respectively.[3,5] The eigenvalues of A are $-1 \pm 2j$ and -3. The CLEs for the initial weighting matrices $Q_{x0} = I_3$ and $R = I_2$ are $\Lambda_0 = [-1.722 \pm 2.176j, -3.540]^T$. The specified CLEs are $\Lambda^* = [-4, -5, -7]^T$. In this case, since the CLEs are shifted from complex conjugate eigenvalues to real ones, shifting is composed of two stages: 1) shifting Λ_0 to $\Lambda' = [-4.5 + j\varepsilon, -4.5 - j\varepsilon, -6]^T$, which are chosen properly on the real axis; and 2) shifting $\Lambda'' = [-4.5 + \varepsilon, -4.5 - \varepsilon, -6]^T$ to Λ^*. In this example, the positive parameter ε smaller than 0.01 is good for satisfactory precision of pole placement.

Figure 1 shows a sketch of the trajectories of the CLEs shifted from Λ_0 to Λ^*. Figure 2 shows trajectories of the diagonal elements of the weighting matrix Q_x obtained by integrating Eq. (28) along the trajectories of the CLEs. The section of 0 through 1 in the horizontal axis corresponds to the first stage, and the section of 1 through 2 corresponds to the second stage. In this example, Q_x is positive definite for all of the CLEs, and the weighting matrix Q^*_x for the desired eigenvalues Λ^* is $Q^*_x = \mathrm{diag}\{7.503, 16.33, 19.67\}$, which is a positive definite matrix. Incidentally, the weighting matrix obtained by Saif's method is not nonnegative definite.

Simulation 2: Lateral-Directional Control of F-4 Aircraft

Let us consider the same design problem as in Refs. 12 and 13. The matrices A and B in Eq. (1) are given by

$$A = \begin{bmatrix} -0.764 & 0.387 & -12.9 & 0 & 0.952 & 6.05 \\ 0.024 & -0.174 & 4.31 & 0 & -1.76 & -0.416 \\ 0.006 & -0.999 & -0.0578 & 0.0369 & 0.0092 & -0.0012 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}^T$$

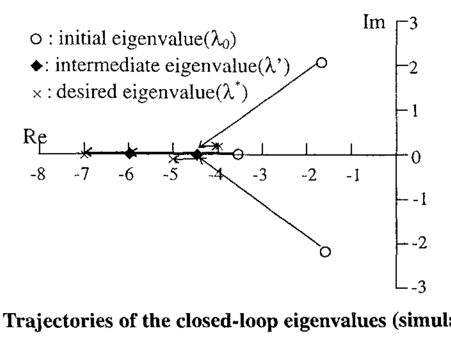

Fig. 1 Trajectories of the closed-loop eigenvalues (simulation 1).

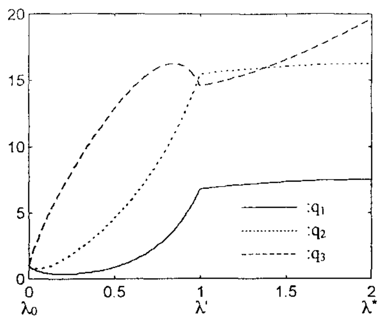

Fig. 2 Trajectories of diagonal elements of the weighting matrix (simulation 1).

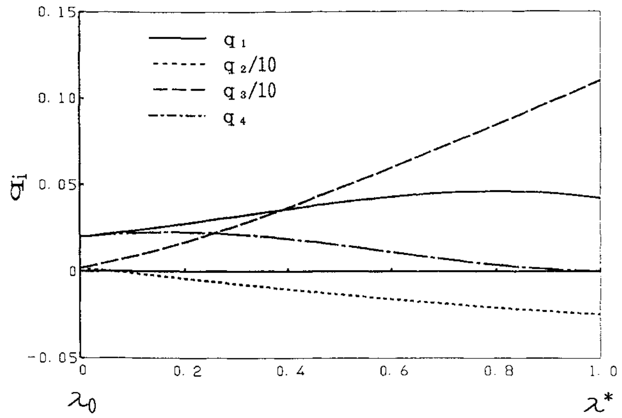

Fig. 3 Trajectories of diagonal elements of the weighting matrix (simulation 2).

The state variable vector is $x = [p, r, \beta, \phi, \delta_r, \delta_a]^T$, and the control variable vector is $u = [\delta_{rc}, \delta_{ac}]^T$. The variables indicate p = roll rate (rad/s), r = yaw rate (rad/s), β = sideslip angle (rad), ϕ = roll angle (rad), δ_r = rudder deflection angle (rad), δ_a = aileron deflection angle (rad), δ_{rc} = rudder command (rad), and δ_{ac} = aileron command (rad). The OLEs are $0.1034 \pm 2.093j$, -0.7828, and -0.006188 for the aircraft dynamics and -10 and -5 for the rudder and aileron actuator dynamics.

Now let us consider assignment of the eigenvalues regarding the aircraft dynamics only, and accordingly choose the output vector as $y = [p, r, \beta, \phi]^T$ or C as $[I_4 \;\; 0]$. Choosing $D = [0 \;\; I_2]$, the C and D matrices make $E = I_6$, which means $\xi = x$. The desired eigenvalues are given as $\Lambda^* = [-0.63 \pm 2.42j, -4, -0.05]^T$. The initial weighting matrix is chosen as $Q_{x0} = \text{block-diag}[Q_{y0}, \mathrm{diag}\{0, 0\}]$, where $Q_{y0} = 0.01 I_4$. The weighting matrix, $R = I_2$, is fixed while shifting the CLEs. The CLEs for (Q_{x0}, R) are $\Lambda_{c0} = [-0.5154 \pm 2.139j, -1.053 \pm 0.2866j]^T$, and $\Lambda_{a0} = [-9.993, -4.855]^T$. Λ_{c0} is shifted to Λ^*. To shift the complex conjugate eigenvalues to the real ones, we have to take the two-stage shifting as in the previous example. More specifically, $-1.053 \pm 0.2866j$ are shifted to a repeated real eigenvalue, and then to -4 and -0.05 on the real axis. As can be seen by drawing a figure of the eigenvalue trajectories, however, it is inevitable to pass through the OLE, -0.7828, irrespective of the selection of the repeated eigenvalue. This problem can be resolved by regarding the closed-loop system for (Q_{x0}, R) as a new open-loop system. Designing an LQR for the new system using the same weighting matrices Q_{x0} and R, then we obtain the new initial CLEs, $\Lambda_{c1} = [-0.6727 \pm 2.141j, -1.453, -1.158]^T$ and $\Lambda_{a1} = [-9.987, -4.685]^T$. Λ_{c1} is shifted to Λ^*. In this case, the two-stage shifting is not necessary. Since the new OLEs are Λ_{c0} and Λ_{a0}, the CLEs do not pass through the OLEs. Integrating Eq. (28) from Λ_{c1} to Λ^* yields $Q^*_{xc1} = \mathrm{diag}\{0.03248, -0.2542, 1.094, -0.009552, 0, 0\}$. By adding Q_{x0} to Q^*_{xc1}, the weighting matrix for the original open-loop system is obtained as $Q^*_x = \mathrm{diag}\{0.04248, -0.2442, 1.104, 0.000448, 0, 0\}$.

Figure 3 shows trajectories of the diagonal elements of Q_x, which is obtained by adding Q_{x0} to Q_{xc1}, whose diagonal elements are solutions of Eq. (28) for the CLEs from Λ_{c1} to Λ^*. The figure

Table 1 Closed-loop eigenvalues and feedback gain matrix

		Results in Ref. 13	
	Proposed method	$\sigma = 1$	$\sigma = 10$
Closed-loop eigenvalues	$-0.63 \pm 2.42j$	$-0.464 \pm 2.35j$	$-0.606 \pm 2.41j$
	-4.0	-3.60	-3.96
	-0.04997	-0.0448	-0.0493
	-10.14	-29.4	-209.0
	-3.007	-11.9	-101.6
Feedback gain matrix	$\begin{bmatrix} 0.03579 & 0.3821 \\ -0.1302 & -0.1963 \end{bmatrix}$	$\begin{bmatrix} -0.5083 & -0.01085 \\ 0.4849 & -0.00354 \end{bmatrix}$	$\begin{bmatrix} -0.06044 & 0.02922 \\ 0.01461 & -0.1249 \end{bmatrix}$

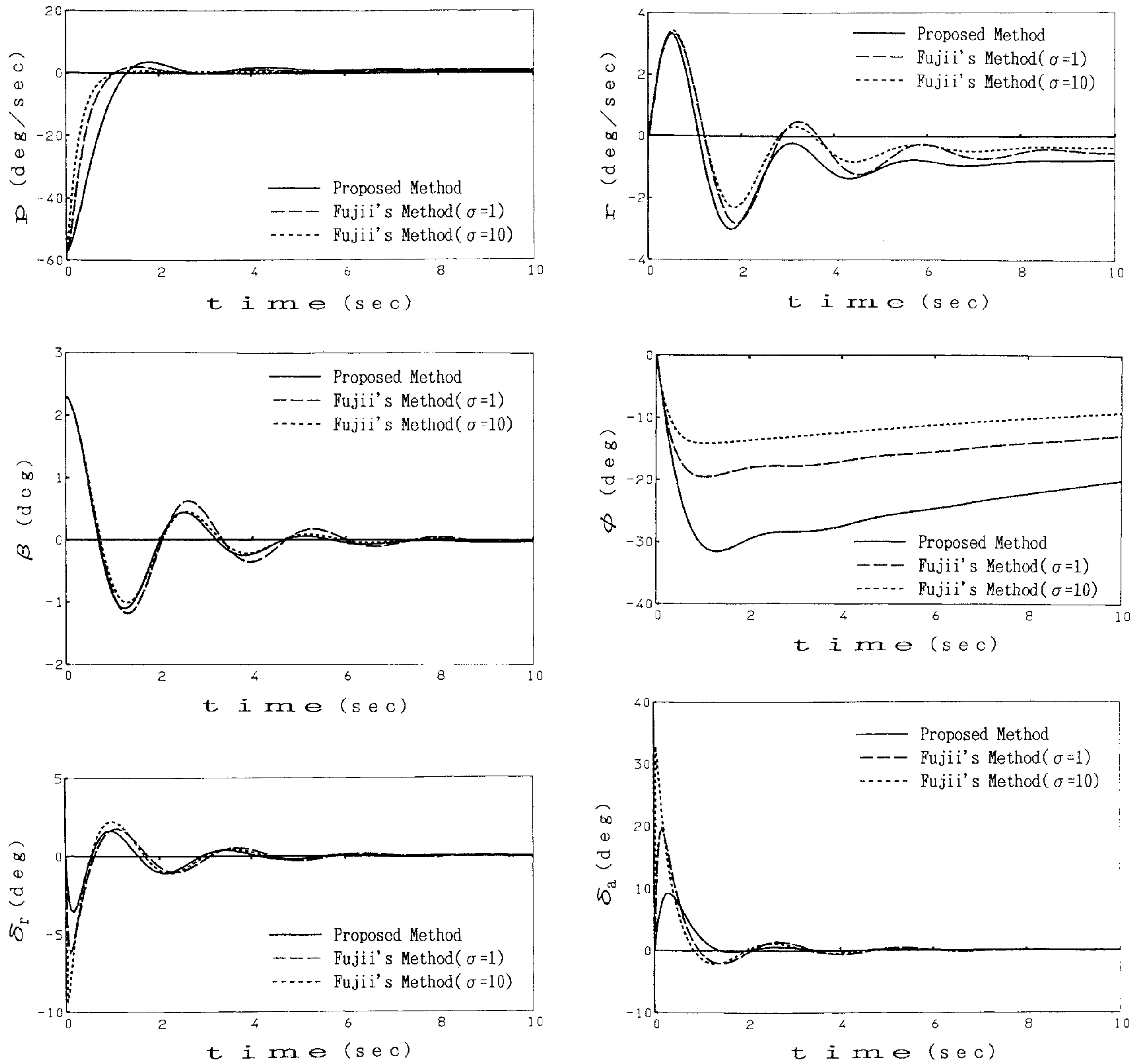

Fig. 4 Time responses of the closed-loop system.

indicates that Q_x is indefinite for the most part of the CLEs on the trajectories and eventually for Λ^*.

Table 1 shows the optimal feedback gain matrix K obtained for Q_x^*, and the eigenvalues of $A + BK$. The results are compared with those in Ref. 13. In the table, σ is a design parameter used in the paper. The method proposed in Ref. 13 gives optimal feedback gains in the sense that the corresponding weighting matrix is nonnegative definite. However, since it is an asymptotic approach, the CLEs for the aircraft dynamics do not exactly agree with the assigned eigenvalues, to which the CLEs converge as $\sigma \to \infty$. This means that the feedback gains can grow too large. In the proposed method, the CLEs for the actuators do not move much, so that the feedback gains obtained are moderate, while achieving exact pole placement. However, the feedback gain matrix is not optimal, since the weighting matrix is indefinite.

Figure 4 shows the time responses of the state variables for the three cases in Table 1. The time responses of the outputs except for the roll angle are similar. Although the method in Ref. 13 gives smaller deviation of the roll angle than the proposed method, the control input is used much more, especially near the initial time, $t = 0$. This is so because the feedback gains are too large, which provides optimality but results in too small eigenvalues for the actuators.

In fact, the command input, whose responses are not shown here, reaches as much as -110 (deg) for the rudder and 380 (deg) for the ailerons in the case of $\sigma = 10$, whereas in the proposed method the command inputs are within ± 10 (deg).

VI. Conclusions

In this paper, we have proposed a method of pole placement that determines a weighting matrix of the LQR by continuous shifting of the closed-loop eigenvalues. The design procedure, as well as the basic idea, presented in this paper is simple and practical. A feature of this method that other methods do not have is that the weighting matrix is obtained as a diagonal one. Such a diagonal weighting matrix makes it clear what weight is put on each state or output, so that it will become easier to understand the behavior of the closed-loop system. Taking advantage of this feature, one can design an optimal output regulator with desired CLEs, where the number of CLEs to be assigned is limited.

On the other hand, the proposed method has some disadvantages. First, the method requires a large amount of computation to solve a set of differential equations. This drawback can be alleviated by choosing root loci from the initial eigenvalues to the desired ones so that the length of the root loci can be small. Second, the design procedure does not guarantee the nonnegative definiteness of the weighting matrix, as pointed out by other researchers like Saif. Therefore the optimality of the regulator is not always guaranteed. Third, restricting the form of the weighting matrix to a diagonal one may limit the area of assignable CLEs for which the optimality condition holds to a smaller part of the complex plane. Fourth, in the output regulation, the behavior of the nonassignable eigenvalues cannot be predicted, which means that those eigenvalues may go into the right half-plane, if the weighting matrix is indefinite. Another problem with the method is that the derivative matrix can become singular in some cases. In numerical computation, the singularity can be avoided, but we must pay some cost; namely, the method requires to shift all of the OLEs to avoid the singularity problem, even if part of the OLEs only are wanted to be changed, and besides trial and error may sometimes be required to find out an initial weighting matrix that prevents the CLEs from passing through the OLEs. For all of these disadvantages, the simulation results show that the proposed method works well in practice and can be a useful design tool for the LQR.

Appendix: Proof that Q_y is Independent of D

Let $S(\lambda)$ defined for the transformed system, Eq. (32), be $S'(\lambda)$, i.e.,

$$S'(\lambda) = (\lambda I_n - A')^{-1} B' R^{-1} B'^T \left(\lambda I_n + A'^T\right)^{-1} \tag{A1}$$

Using the definition of A' and B', Eq. (A1) can be rewritten as

$$S'(\lambda) = E S(\lambda) E^T \tag{A2}$$

Then the characteristic equation Eq. (8) becomes

$$\det[I_n - Q'_x S'(\lambda)] = \det\left[I_m - Q_y C S(\lambda) C^T\right] \tag{A3}$$

Since Eq. (A3) does not include D, Q_y does not depend on D.

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